On inductive and deductive reasoning

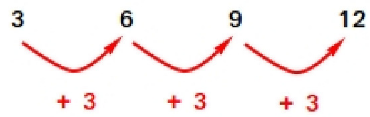
Table of content

1. Inductive and deductive reasoning in mathematics

Let us start with some examples which illustrate inductive and deductive reasoning in mathematics. As you read these examples see if you can explain the difference between inductive and deductive reasoning, and see if you can define inductive and deductive reasoning.

1.1. Example 1: Patterns in number sequences

The diagram below shows a sequence of numbers. There seems to be a pattern as to how we get from one number to the next. So, let us suppose that in order to obtain the next number we add 3. This implies that the fifth number should be 15.



Pattern observed: Add 3 to the previous term.

Conjecture: The next number is 15.

If we use the notation N(k) to mean "the number in the k^{th} position" we have that N(1) = 3, N(2) = 6, N(3) = 9, etc. We could then say that, for any number in the k^{th} position, the general pattern is N(k+1) = N(k) + 3, where we start at N(1) = 3. Is this an example of inductive reasoning or deductive reasoning?

1.2. Example 2: Statements about arithmetic

Consider the following statements, and decide whether the statements are inductive or deductive:

1. "Even numbers differ from odd numbers by 1. So, if we add 1 to the number 3 we obtain an even number;
2. "If $x = 3$ then $2x = 6$";
3. "An even number is an odd number plus 1. So, adding two odd number gives an even number plus 2.";

Hint: For 1. and 3. consider each sentence separately. For example, in 3. is "An even number is an odd number plus 1" an inductive or deductive statement? Is "So, adding two odd number gives an even number plus 2" an inductive or deductive statement? (look at the language of the second sentence to help you answer this).

1.3. Example 3: More patterns in number sequences

We have to be careful of inductive reasoning in mathematics. For example, not all formulae are true in general. Consider Euler's formula for supposedly generating prime numbers: $n^2 - n + 41$. Starting with $n = 1$ we can supposedly generate prime numbers. A few of these are shown below:

$\boldsymbol{n}$	$\boldsymbol{n^2 - n + 41}$	**Prime?**
1	41	**Yes**
2	43	**Yes**
3	47	**Yes**
4	53	**Yes**
.	.	.
.	.	.
.	.	.
38	1447	**Yes**
39	1523	**Yes**
40	1601	**Yes**

All the numbers in the middle column are prime numbers. So it seems that the formula $f(n) = n^2 - n + 41$ generates prime numbers. This is a deduction. But is it true that the formula generate all prime number? The answer is no, because when $n = 41$ we have

$$f(41) = (41)^2 - 41 + 41 = 41^2 \ ,$$

i.e. $f(41) = 41 \times 41$ which is not prime because it is divisible by 41. What does this example suggest about the effectiveness of reasoning by induction from specific cases?

1.4. Example 4: Proof by induction

The classic example of induction in mathematics is proof by induction. In induction we try to generalise from specific cases. In maths this means that we try to find a general formula which generates not only the specific cases we already have, but will then correctly generates all other possible cases, namely an infinite number of cases. But since we can't generate an infinite number of cases we have to generalise the method of confirming the truth of a formula. such a method is proof by induction.

For example, given a formula, we assume that the formula works for one specific value n (say $n = 1$, although this is not always the case), then we prove that if it works for some other specific integer value, say k, then the formula will also give the correct answer for the next integer value $k + 1$.

So, let us prove that the sum of integers up to n is given by $n(n+1)/2$. In other words, we want to prove

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

The left-hand side of this equation is simply

$$1 + 2 + 3 + 4 = 5 + \cdots + n.$$

The right-hand side is a closed formula which is supposed to allows us to calculate this sum. The aim is to prove that this formula works for all integer values of n.

To do this, let $P(n)$ represent the above formula. The first step is to show that $P(1)$ is true. In other words, we want to show that the equation above works for at least one value of n. To show this we need to consider the left-hand side and right-hand side separately. So

- Set $n = 1$ in the left-hand side: $\sum_{i=1}^{1} i = 1$,
- Set $n = 1$ in the right-hand side: $1(1+1)/2 = 1$.

Hence $P(1)$ is true. Now we want to show that if $P(k)$ is true for any value k, then this will lead to $P(k+1)$ also being true. Hence if $P(k)$ is given as

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

we want to show that we can obtain $P(k+1)$ which is given as

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}.$$

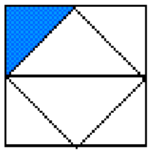
This is indeed the case (I won't do the maths here). It is because we can prove that $P(k+1)$ follows from $P(k)$ that we can say that, in general, $P(n)$ is true for all integer values n, i.e. that the right-hand side formula will always work.

This is a case of inductive reasoning because we will have shown that the right-hand side formula generates the next number along in the sequence. So if we have proved that $P(1)$ is true then the formula will correctly generate $P(2)$. Since $P(2)$ is now true the formula will correctly generate $P(3)$. Since $P(3)$ is now true the formula will correctly generate $P(4)$, etc.

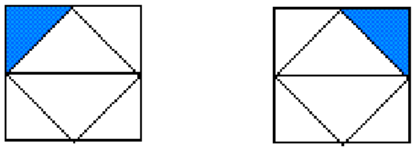
1.5. Exercises

1) Look at the sets of shapes below and conjecture the shape you would expect for set four.

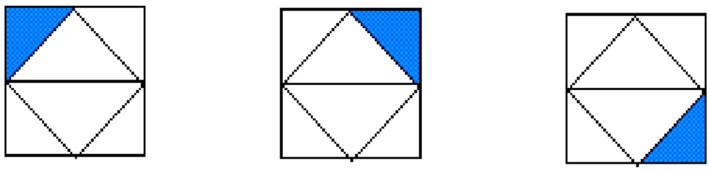
i) set one



ii) set two



iii) set three



iv) set four — From the three examples above identify the next set of shapes (inductive reasoning).
Generalise this reason: Write down a description of the pattern for generating sets of shapes (inductive reasoning).

v) set five — Apply the general understanding you developed in iv) to find the specific answer for set five (deductive reasoning).

2) Look at the sets of numbers in the grids below and conjecture the numbers in set iv).

Set i)

1	
1	1

Set ii)

1		
1	1	
1	2	1

Set iii)

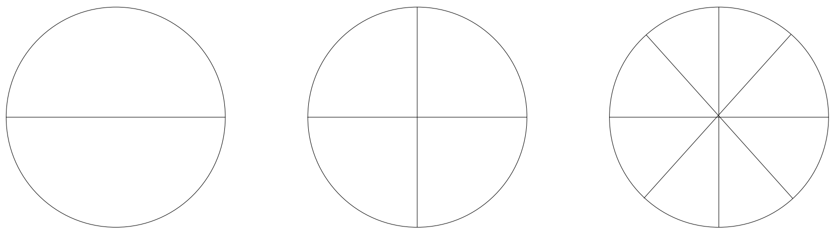
1				
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1

Set iv)

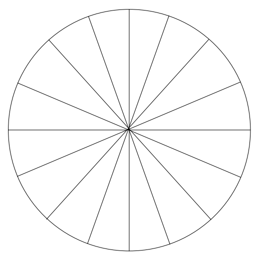
2. Commentary of induction and deduction as it relates to mathematics

Inductive and deductive reasoning can be thought of as opposite types of reasoning. Both involve the process of moving from something we know to something we don't know. In induction we take specific cases or examples and attempt to find a general law or pattern. For example, consider the problem below.

– Specific examples:

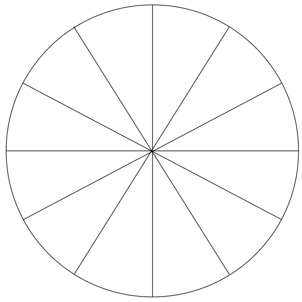


– Inductive assumption:
The next diagram is …



– Supposed general pattern: The number of lines within the circle doubles every time

However, the next diagram could have been



i.e. our inductive assumption could have been, “Add two more line inside the circle”.

As another example consider the sequence

$x =$	1	2	3	4
$y =$	12	12	12	18

What would you guess for the value of y when $x = 4$? You might guess $y = 12$, and this would be an example of inductive thinking. But I set up this example to produce the answer $y = 18$, so you were not correct in your guess. How can $y = 18$? Because I was using the formula

$$y = (x - 1)(x - 2)(x - 3) + 12 .$$

So, in induction we conjecture the general pattern based on a sequence of individual cases, and we then use this general pattern to find what we believe will be the next number. From the above example we see that inductive reasoning can lead us to the wrong conclusion.

However, deductive reasoning is reasoning which starts with statements or theories which are true in general and from which we derive specific examples or cases. These specific cases or examples are automatically true since they represent particular illustrations of the general statement or theory. For example, if I give you the formula

$$y = (x - 1)(x - 2)(x - 3) + 12$$

(which represents the general law/pattern) you can then deduce the sequence of numbers to be 12, 12, 12, 18, 36, 72, etc.

Another example of deductive reasoning is

- General pattern/law: Numbers ending with 0 and 5 are divisible by 5.
- Specific example: The number 20 ends with 0.
- Deductive conclusion: The number 20 is divisible by 5.

Yet another example: given the general statement

$$(a+b)^2 = a^2 + 2ab + b^2$$

then it is true that

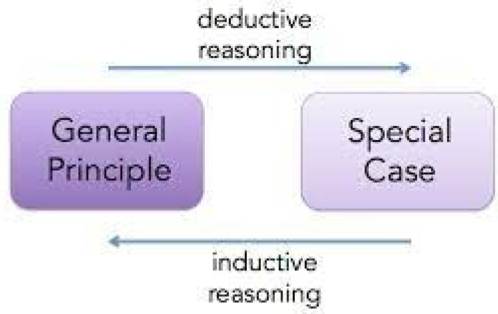
$$103^2 = (100+3)^2 = 100 \times 100 + 2 \times 100 \times 3 + 3 \times 3 .$$

All of arithmetic and algebra is deductive:

- Consider the statement “adding 1 to an odd number gives an even number”. Then a specific example of this is $3 + 1 = 4$. This is an example of deduction, where we have shown a specific example based on the general statement;
- Solving $x - y = 0$ and $x + y = 2$ requires deductive reasoning in the algebra we perform;
- Given that $a^n . a^m = a^{n+m}$ we have that $3^4 . 3^{23} = 3^{27}$;
- All numbers ending in 0 or 5 are divisible by 5. The number 25 ends in a 5 so it is divisible by 5;

Mathematics is a deductive discipline. It starts by stating axioms, which are statements which are considered self-evident and do not need to be proved (some people say that these axiom cannot be proved, which is why we need to take them as axioms). These axioms are therefore taken as true. From the axioms we deduce other mathematical statements which we then prove to be true. These true statements are then called theorems. So, in mathematics, provided the general statement pattern or principle is true, deductive reasoning will always lead us to the correct conclusion. This is not the case for the physical sciences.

We can interpret inductive reasoning as a "Bottom-Up" approach since we use specific cases to find an underlying pattern so that we can draw a general conclusion. On the other hand, deductive reasoning can be called the "Top-Down" approach as its draws specific conclusions from generalized patterns or laws (which are known to be true).



3. Inductive and deductive reasoning in the physical sciences

3.1. Example 1: The structure of the solar system

An example of inductive reasoning: At one time it was believed that the Earth was stationary at the centre of our planetary system, and that all the other planets and the sun revolved around the Earth in circular orbits. This is inductive reasoning because from the individual case that the Earth did not move in the sky (because the stars remained fixed in the sky) and the cases of Venus, Mars, Jupiter, and the Sun moving in a circular manner across the sky people derived the general principle that the Earth was the centre of the planetary system, and every other planet and the sun revolved about the Earth.

However, this belief caused problems because the planets were not always in the position predicted by such a theory. So, in order to account for the actual positions of the planets a complicated system of circular motion within circular motion (called epicycles) was developed. This system is credited to Claudius Ptolemy (c. 100 – 170AD). He developed this model in order to be able to keep this circular, Earth centred theory of motion.

Johannes Kepler (1571 – 1630) was the first person to show mathematically that planets revolve around the sun, and do so not in circular orbits but in elliptical orbits. His laws of motion were also derived inductively in the sense that he used observations of planetary position and motion across the sky (just like the Ptolemaic model did). However, the reason Kepler ended up finding the correct movement of the planets was because he didn't just use a few observations of planetary position but a huge amount of observations (all gathered by his teacher Tycho Brahe (1546 – 1601)). As a result of so many accurate observations by Tycho Brahe, Kepler was able to "interpolate" the path the planets had to take in order to correctly predict their positions at any given time.

3.2. Example 2: The discovery of Uranus and Saturn

An example of deductive reasoning: Newton's theory of gravity was/is a general theory which explains (amongst other things) the motion of the planets. The theory had been consistently shown to be correct because it was able to predict the position of known planets to a high degree of accuracy (Newton's theory is so good enough that it was used as the underlying maths that sent man to the moon). However, there were minor errors in the predicted positions compared to the actual position of the known planets, the outermost known planet being Saturn.

Two possibilities could explain the errors: i) Newton's laws were wrong, or ii) there was another planet beyond Saturn which was influencing Saturn's position. Item i) was rejected since Newton' theory kept giving very accurate predictions. So people believed item ii) was more likely.

So, it was deduced that, because the position of Saturn wasn't exactly where it should have been, there must be a planet beyond Saturn's orbit whose gravitational effect was influencing Saturn. As a result, people started looking for this new planet in the position predicted by Newton's theory. And this new planet was indeed found in March 1781 by William Herschel (1738 – 1822). This planet was then called Uranus. This is an example of deduction. From a general theory now accepted to be correct people deduced that there had to be another individual planet beyond Saturn which could explain Saturn's anomalous behaviour.

So it was that Newton's theory of gravitation was again shown to be correct. Then when it was discovered that Uranus was in a position not exactly predicted by theory. So people thought that there must be another individual planet influencing the orbit of Uranus. And indeed such a planet was discovered by Johann Galle (1812 – 1910), a German astronomer, using calculation by Urbain Verrier (1811 – 1877) a French astronomer and mathematician. The planet was called Neptune, and again the validity of Newton's theory of gravitation was confirmed.

4. Commentary on inductive and deductive reasoning for the physical sciences

In science inductive reasoning is that reasoning which makes use of patterns, trends, specific observation and examples to draw a general conclusion or hypothesis. The previous examples which illustrated inductive reasoning could be called *induction by enumeration* because each example made the use of specific numbers or data listed in some kind of order. It was then this ordering which allowed us to identify a pattern.

We clearly need to be careful when using inductive reasoning, as illustrated by the examples of Claudius Ptolemy and Johannes Kepler. It is not enough to have a few pieces of data if we are going to derive a general pattern (say, circular orbits around the Earth) to the behaviour of a phenomena. We may need a lot of data in order to develop a correct general pattern (say, elliptical orbits around the Sun). It was *the mass of data accumulate* by Tycho Brahe which allowed Kepler to use the data as evidence which supported a Sun-centred elliptical orbit theory for the motion of planets.

One way to justify inductive reasoning, and therefore to confirm a hypothesis developed inductively, is to compare the weight of evidence in favour of the claim against other alternative/incompatible hypotheses. We might call this a *comparative confirmation of inductive reasoning.*

We have just seen an example of this in terms of the Earth-centred versus Sun-centred planetary system. The analysis of the data by Kepler made it much more likely that planets travelled around the Sun in elliptical orbits than around the Earth in circular orbits. Theoretically speaking it is possible that, as we gather yet more data about how planets revolve around a common centre, the Sun-centred theory may be overthrown by a more accurate theory. However, this will not happen because

- we have now gathered so much data over the last 100-150 years,
- we have sent satellites into space based on the mathematics of a Sun-centred system,
- we have sent man to the moon based on the mathematics of a Sun-centred system,

etc. that there is no doubt we live in a Sun-centred system.

So in the case of a hypothesis developed inductively the simplest kind of alternative hypothesis we can use for comparison is one which is the opposite of our hypothesis:

- "The Earth is at the centre of our planetary system" versus "The Sun is at the centre of our planetary system";
- "The planets move in circular orbits" versus "The planet move in elliptical orbits";

However, deductive reasoning is reasoning which starts with statements or theories which are true in general and from which we can derive specific examples or cases. These specific cases or examples represent particular illustrations of the general statement or theory. Examples of deduction in the physical sciences include:

- All noble gases are stable. Neon is a noble gas. Hence Neon is stable (this particular form of thinking is called a syllogism);
- One of Newton's laws of motion is $F = ma$ where F is the force applied to an object, m is the mass of the object and a is the acceleration of the object in the same direction as the force. From this general law we can say that if a car is travelling on a road from left to right and is slowing down (i.e. decelerating) there must be a force acting on the car which acts from right to left (for example, wind, the brakes of the car, etc.);
- By using Newton's law of gravitation it was possible to deduce that there must have been a planet beyond Saturn which was affecting Saturn's orbit. This planet was found and was called Uranus. Again, by using Newton's law of gravitation it was possible to deduce that there must have been a planet beyond Uranus which was affecting the orbit of Uranus. This planet was found and was called Neptune.

Deduction in the physical sciences is based on a general theory about a phenomenon. How do we obtain such a general theory? Sometimes

- by induction, by conducting experiments in order to collect specific data from which we can identify a general pattern (so called empirical laws such as Newton's laws of motion, Kepler's laws of planetary motion, etc.),

or

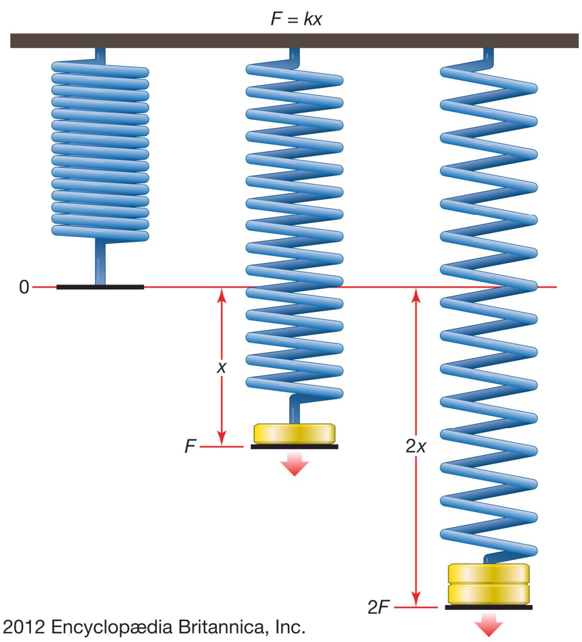
- by theoretical mathematical analysis of how we believe a physical phenomenon should behave. This is what is done in theoretical physics. For example, Einstein's theory of relativity was deductively developed (i.e. by mathematical analysis). He had no experimental data about gravity on which to base his general theory. It was then the case that his theory predicted certain particular phenomena such as the bending of starlight and the expansion of the universe. These specific cases were then confirmed experimentally.

A deductive conclusion is always true if the premise is true. However, an inductive conclusion may or may not be true because you don't know for sure whether the pattern you have identified is actually the general pattern (see the example of the lines inside a circles of p5-6 above).

5. The interplay between inductive and deductive reasoning

There is an interplay between inductive and deductive reasoning. For example, in 1676 Robert Hooke (1635 – 1703), an English physicist and mathematician, was able to develop a general mathematical relationship which described how the stretch in a spring was related to the weight at the end of the spring. Obviously the heavier the weight the greater the stretch, but if we know how much a spring has stretched with a 1kg weight, what is the effect on stretching for a 2kg weight? Hooke found that the amount of stretch is proportional to the weight (the stretch also depends on the type of material the spring is made of. Different materials have different springiness and therefore different stretchability).

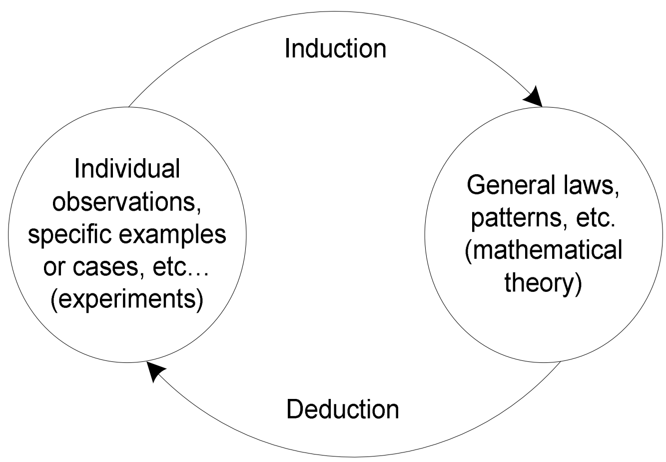
So the question is, If we double the weight do we double the amount by which the spring has stretched from its unstretched length? If we triple the weight do we triple the amount by which the spring has stretched from its unstretched length? Etc.



So, we come to the following

- *Inductive reasoning leading to a general law*: By performing many experiments using different weights on the same spring, and using the same weight on different springs we are able to record data which represents the different amounts of stretching by springs. We can then find the general pattern of stretchiness and express this as the formula Hooke devised to be $F = -kx$. Here F is the force from the weight (in kg) attached to the end of the spring, x is the amount of stretch (in cm or m) that has occurred and k is a number which describes the inherent stretchability of the material which the spring is made of (i.e. k is different if the spring is rubber or metal or plastic, etc.).
- *Deductive reasoning from the general law*: Now that we know Hooke's law we can apply it to all sorts of other objects where a weight, or other pulling power, is applied. In that case we then use Hooke's law to predict the amount of stretchiness of an object under a given force. We can predict the stretching and/or compression of
    - a rope used by a car to tow a caravan;
    - a car suspension system, or even of the tyres of the car, as the car travels along a road. Such travelling causes the car to "bounce" which is what causes the compression or stretch of the suspension system or the tyres;
    - the spring in a retractable pen;
    - a balloon when someone blows air into it. Then, when the balloon is evacuated, it shrinks in size. The expansion and compression of the balloon depend on the force with which the air is pressed into it;

In these, and other cases, we would conduct experiments to collect specific data which would confirm the predictions made by the general law.



So, if a general law or principle is known about a physical phenomenon, experimentation leads to deduction about the phenomena: specific properties, behaviour, etc. On the other hand, if no general law or pattern is known we start with experimentation in order to observe how that phenomenon behaves. A mass of data is collected in order to understand more accurately the behaviour of this phenomenon, until a pattern begins to emerge or a mathematical theory can be developed which matches the data sufficiently well enough. So, we are inferring the general pattern or mathematical theory from the specific data. This is induction.

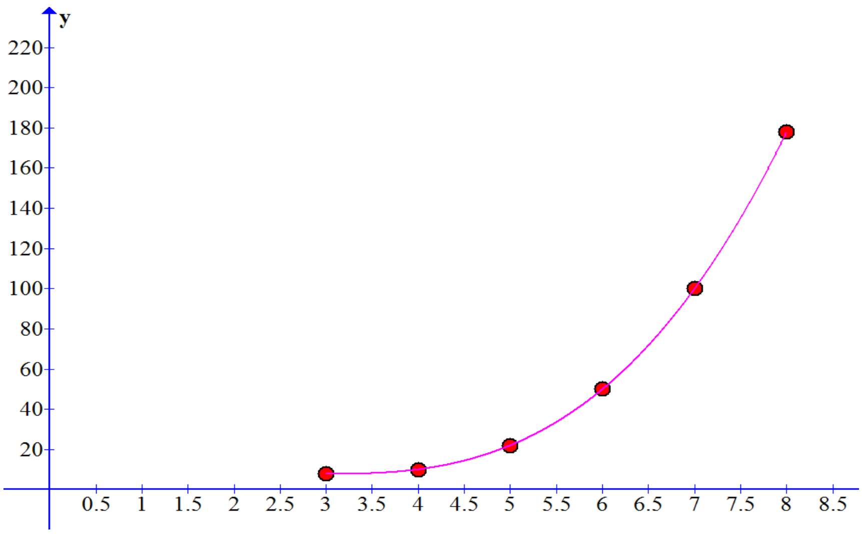
It can be said that inductive reasoning allows you to learn something new about the world whilst deductive reasoning allows you to apply what you have learnt.

6. More on inductive reasoning in mathematics: The case of statistics

Statistics deals with probabilities and trends in data not with definite exact relationships. For example, data and graph (i) below illustrates an exact relationship between the *x* data and the *y* data. But data and graph (ii) does not. It only suggests a trend. So, no exact answer will be obtained when using methods for analysing trend data.

x	*y*
3	8
4	10
5	22
6	50
7	100
8	178

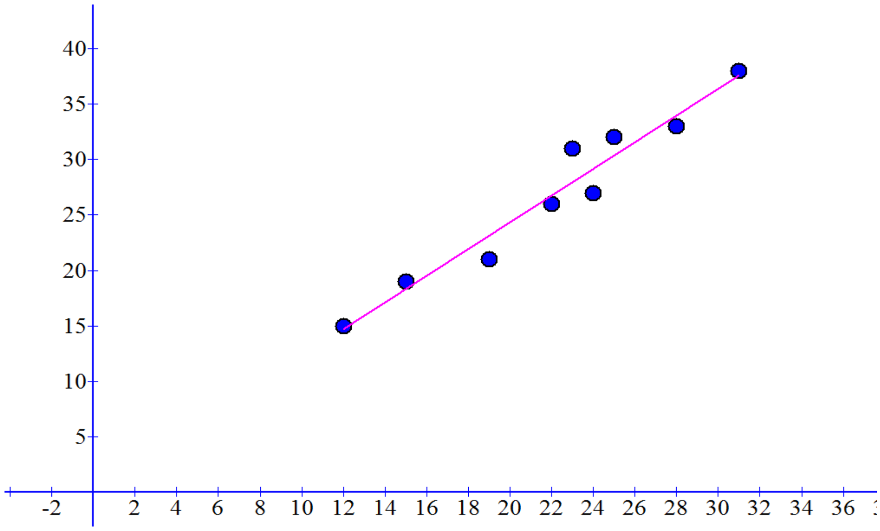
Data (i)



Graph (i)

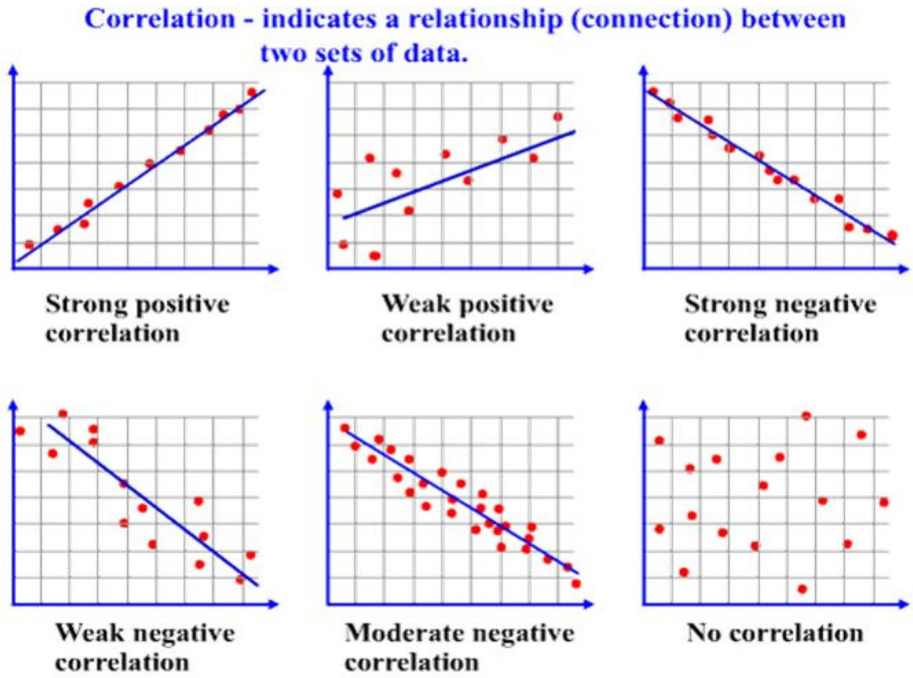
x	y
12	15
22	26
19	21
15	19
31	38
25	32
28	33
24	27
23	31

Data (ii)



Graph (ii)

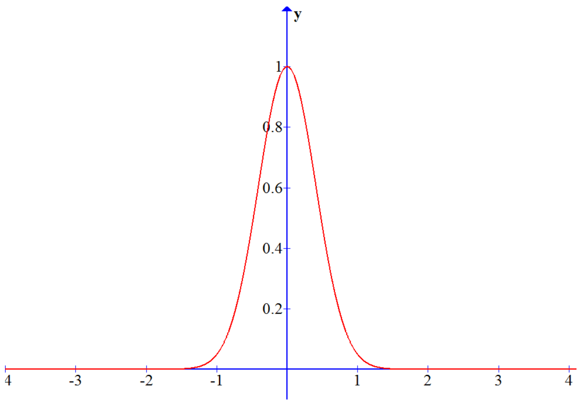
Some trends are strong, some are weak as illustrated below (the strength of a trend is called correlation).



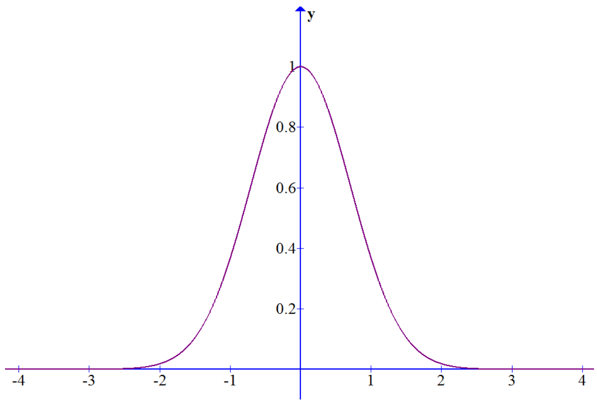
Factors which determine the answers you obtain in such situations include

- how you collected the data, i.e. the statistical experimental procedure you used;
- where and when you collected the data, i.e. is location or time of day important when collecting data, or was there anything influencing the ability to collect good quality data,
- how much data you collect, i.e. too little data will not allow you to see an accurate trend.

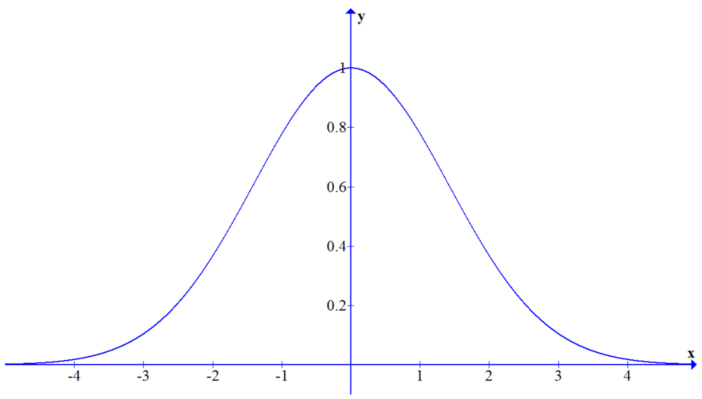
A lot of statistical methods are based on the mean and standard deviation (spread) of the data collected. The mean is the usual way we calculate the average value of a set of data, and the standard deviation is how spread out the data is from the mean. This is illustrated below for the example of heights of adults in three different countries where the vertical axis represents the mean height of adults in each country.



Spread of heights of adults in the UK



Spread of heights of adults in France



Spread of heights of adults in Germany

So if we want to collect data about the heights of adults in a country we can't go measuring the height of all adults in that country. We have to take a sample of adults, say 1000, people, and then base our decisions about the population on the statistical analysis of this sample alone.

One type of study that can be performed is the study of the average heights of adults. But we can't find the average height of people in the whole country. So instead we take a sample, say of 10 people, and find the average height of this sample. Any hypothesis we develop based on our sample data will be based on inductive reasoning. In other words, we would be trying to make a decision about the average heights of adults which applied to the whole population (in other words, a general decision) based only on 10 specific data values.

But now we have a problem: which 10 people do we choose that will give us the average height closest to the average height of the population? We don't know. So we choose a *random sample* of 100 people. And different random samples will give different means.

To see this suppose the data below represents the population of a country. There are only 100 people in this country, and their heights are recorded in metres.

2.15	1.68	1.67	1.78	2.04	2.13	1.53	1.65	1.49	1.90
1.97	1.90	1.82	2.06	1.71	1.39	1.68	1.91	1.82	2.16
1.83	1.57	1.60	1.38	1.88	1.89	1.33	1.60	1.58	1.19
1.76	1.65	1.99	1.64	1.67	1.69	1.96	2.00	1.59	1.99
1.61	1.96	2.06	1.87	1.80	1.94	1.53	1.95	1.73	1.84
1.44	1.62	1.84	1.78	1.82	1.81	1.81	1.74	2.10	1.85
1.73	1.80	2.06	1.80	1.79	2.00	1.76	1.59	1.81	1.60
1.92	1.68	1.98	1.73	1.88	1.80	1.90	1.78	1.79	1.87
1.77	1.98	1.87	1.63	2.09	1.49	1.63	2.04	1.74	1.78
1.85	1.58	2.14	1.91	1.80	1.55	2.22	1.82	1.74	1.74

The mean height of the population (all 100 people) is 1.79m and the spread is 19.5cm. If we now take the 1^{st} row of the table above as a sample of 10 people we get a mean height of 1.80m and a spread of 24.2cm. If we take a different sample of 10 people, such as row 2, we will get a different mean and spread from sample 1.

In fact if we take *any* sample of 10 people form the population of 100 people the means and spread will be different for all of them, as shown in the table below where I have calculated the mean and spread for each of rows 1 to 5 in the table above as my sample of 10 people.

Sample of 10 people	**Mean (m)**	**Spread (m) (called Standard Deviation)**
Sample 1 (row 1 of table 1)	1.80	0.242
Sample 2 (row 2 of table 1)	1.84	0.217
Sample 3 (row 3 of table 1)	1.59	0.236
Sample 4 (row 4 of table 1)	1.79	0.169
Sample 5 (row 5 of table 1)	1.83	0.167

Note that the mean of sample 4 is the same as the population mean. It is just luck that I chose a sample that had the same mean as that of the population (and even then we wouldn't know that this mean was the same as that of the population). Also note that the spreads of each sample are quite different from the population spread of 19.5cm.

In general, statistics is about making inferences about the population based on samples we have collected. And inferences are based on inductive reasoning. So we will never obtain the exact same results from experiments which use statistical analyses. In the strict sense these experiments are not replicable. This is something we have to live with. Does this mean that the experiment has been falsified? No. Statistics is a useful method of analysis since it provides information about large/big data. We mitigate errors and difference in results by taking large samples, or taking multiple samples. In doing this we reduce the difference/error in results between two experiments conducted independently of each other. Statistics is then about obtaining consistent results within an error margin deemed acceptable.

Exercises

1) "I have a triangle. Triangles are formed by joining three points with straight lines. Therefore any three points allows me to draw a triangle". Is this statement true? Have you used inductive or deductive reasoning here?

2) If you know enough statistic you can determine whether or not statistic is mainly inductive or deductive. For example, is the calculation of significance tests (calculating a *p*-value) or confidence intervals an inductive or deductive method?

3) Describe aspects of your discipline (undergraduate or postgraduate) where induction and deductive reasoning was/is used. For example, were the experiments you did in the lab designed to allowed inductively based conclusions or deductively based conclusions?